

# Edge-Based Viscous Method for Mixed-Element Node-Centered Finite-Volume Solvers

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**AIAA AVIATION Forum and Exposition**  
Session: CFD-22, Convergence Acceleration  
June 30, 2022





- **Edge-based finite-volume methods are widely used**
  - inviscid and viscous fluxes computed in loops over edges
  - offer advantages of efficiency and generality
  - typically, results in discretization stencils extended beyond nearest neighbors
- **Cell-based viscous (CBV) method**
  - viscous fluxes computed in loops over cells
  - used in finite-volume FUN3D flow solver, extensively verified and validated
  - compact nearest-neighbor stencil
  - equivalent to finite-element Galerkin method on tetrahedra
- **Edge-based viscous (EBV) method**
  - maintains compact nearest-neighbor stencil
  - accuracy is similar to CBV method
  - speeds up viscous-kernel computations (viscous fluxes, diffusion, Jacobian)
  - reduces Jacobian size and speeds up linear solver for mixed-element grids
  - six EBV coefficients stored for each interior edge (nine for each boundary edge)
  - EBV coefficients represent grid metrics, do not depend on solution, can be precomputed



# Background: EBV on Tetrahedra



- **EBV method follows approach introduced by Barth** <sup>1</sup>
- **EBV for Reynolds-averaged Navier-Stokes (RANS) equations** <sup>2,3</sup>
  - EBV and CBV solutions and convergence per iteration are similar
  - EBV fraction of viscous kernel is less than 10% of all computations
  - EBV speedup is up to 30% (EBV speedup factor 1.4)
- **EBV formal 2<sup>nd</sup>-order accuracy established** <sup>4</sup>
  - Source terms are introduced for momentum and energy conservation residuals

$$\text{EBV speedup} = \frac{(\text{CBV time}) - (\text{EBV time})}{(\text{CBV time})} \times 100\%$$

$$\text{EBV speedup factor} = \frac{(\text{CBV time})}{(\text{EBV time})}$$

<sup>1</sup>Barth T. J., "Numerical Aspects of Computing Viscous High Reynolds Number Flows on Unstructured Meshes," AIAA 91-0721

<sup>2</sup>Liu Y., et. al., "Edge-Based Viscous Method for Node-Centered Formulations," AIAA 2021-2728

<sup>3</sup>Liu Y., et. al., "Edge-Based Viscous Method for Node-Centered Finite-Volume Formulations," AIAA Journal (under review)

<sup>4</sup>Diskin B., et. al., "Analysis of Edge-Based Method for Diffusion," ICCFD-11; 11-15 July 2022 (accepted)



- **Extended set of EBV edges**

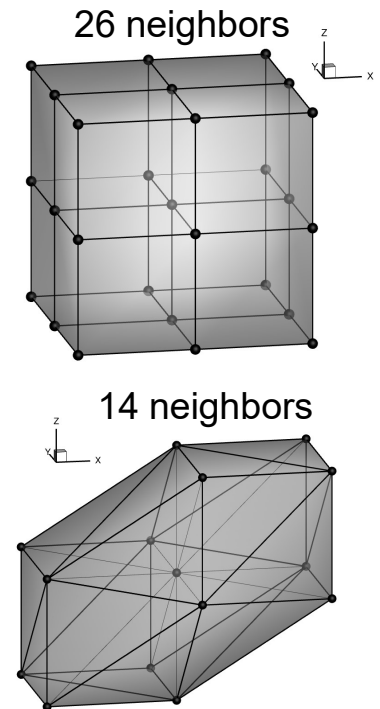
- EBV requires any two vertices of a cell to be connected by edge
- primal edges do not connect some vertices of nontetrahedral cells
- nontetrahedral cells are divided into conformal tetrahedra
- EBV edges include all edges of derived tetrahedral grid

- **EBV implementation on mixed-element grids**

- inviscid fluxes are computed in a loop over primal edges
- EBV fluxes are computed in a loop over EBV edges
- EBV source terms are added at grid points

- **Memory saving and linear solver speedup**

- EBV stencil can be significantly smaller than CBV stencil on primal grids
  - ✓ CBV: ~26 neighbors on hexahedral grids; ~20 - on prismatic grids; ~14 - on tetrahedral grids
  - ✓ EBV: ~14 neighbors on all grids
- reduced Jacobian results in EBV memory savings and EBV speedup of linear solver





- **Discretization scheme**

- node-centered scheme on general unstructured mixed-element grids
- dual volumes around grid points
- meanflow inviscid fluxes:
  - ✓ 2nd-order UMUSCL reconstruction to edge median
  - ✓ Roe's approximate Riemann solver
- CBV and EBV methods for meanflow viscous fluxes and diffusion for turbulence and chemistry models
  - ✓ CBV: Green-Gauss cell-based gradients, augmented by edge derivatives at nontetrahedral cells
  - ✓ EBV: Galerkin on derived tetrahedra expressed as sum of edge-based solution differences
- 1<sup>st</sup> order convection for turbulence model

- **Iterative solvers**

- baseline iterative solver
  - ✓ approximate Jacobian, prescribed CFL, 30 point-implicit multicolor Gauss-Seidel sweeps
  - ✓ Jacobian updates are scheduled according to nonlinear residual convergence
- hierarchical adaptive nonlinear iteration method (HANIM) <sup>5</sup>
  - ✓ strong nonlinear Newton-Krylov solver with solution-adaptive CFL
  - ✓ improves efficiency and robustness

<sup>5</sup>Wang, L., et. al., "Improvements in Iterative Convergence of FUN3D Solutions," AIAA 2021-0857



# RANS Solutions on Prismatic-Hexahedral Grids

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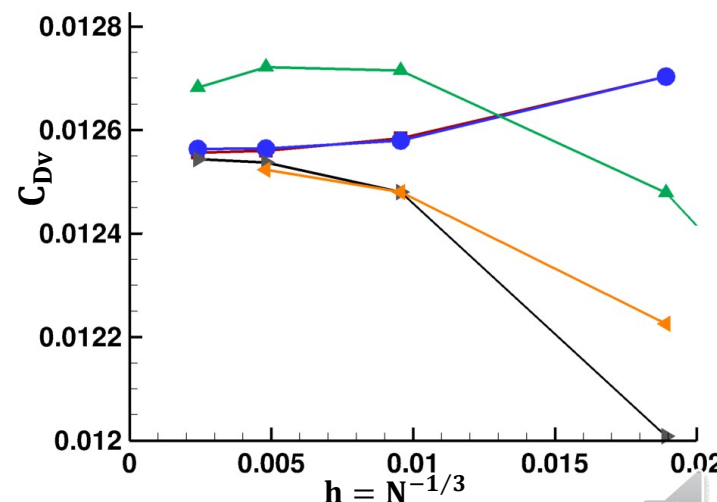
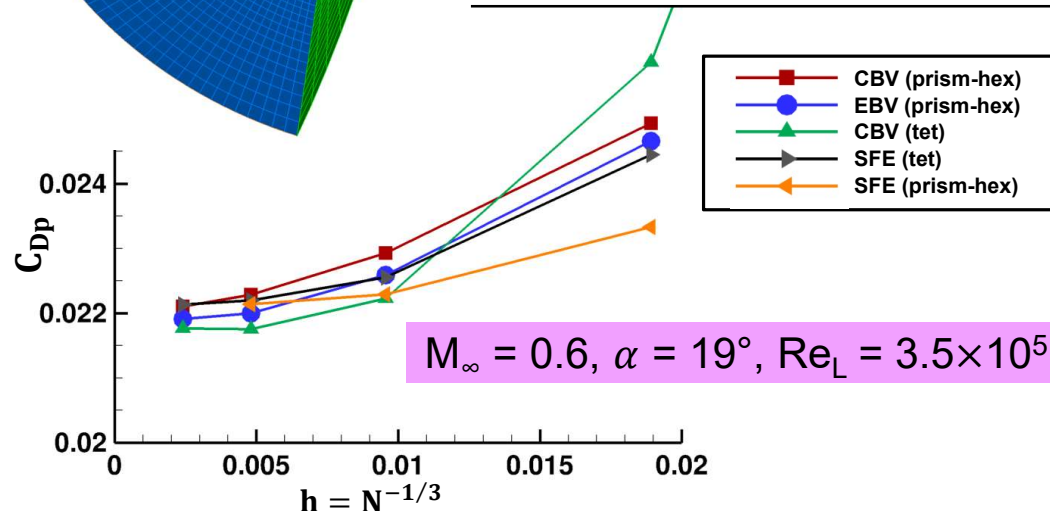
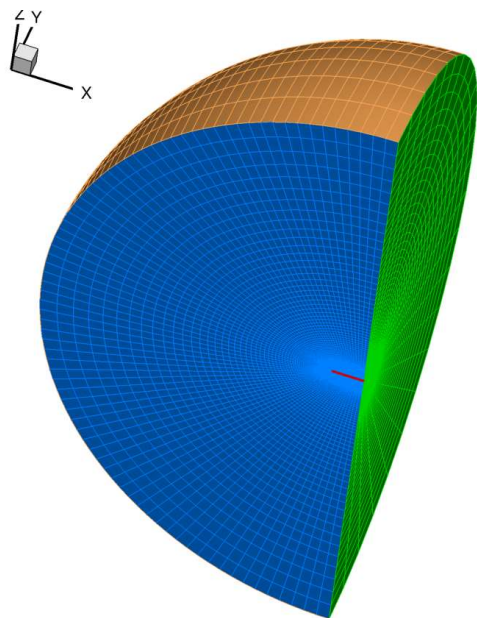


NASA Turbulence Modeling Resource<sup>§</sup> benchmark: separated flow around hemisphere cylinder

RANS with negative Spalart-Allmaras (SA-neg) turbulence model

Family of prismatic-hexahedral grids for hemisphere-cylinder configuration

Grid	Points	Hexahedra	Prisms	Primal Edges	Virtual Edges	CPU cores
PH0	71,368,353	62,914,560	15,728,640	221,384,352	275,832,832	400
PH1	8,995,153	7,864,320	1,966,080	27,821,392	34,551,808	320
PH2	1,143,081	983,040	245,760	3,514,920	4,337,152	40
PH3	147,637	122,880	30,720	448,756	546,688	5



<sup>§</sup>[https://turbmodels.larc.nasa.gov/hc3dnumericspart2\\_val.html](https://turbmodels.larc.nasa.gov/hc3dnumericspart2_val.html). Accessed April 11, 2022





# Baseline-Solver Iterations on PH1 Grid (9M Grid Points)

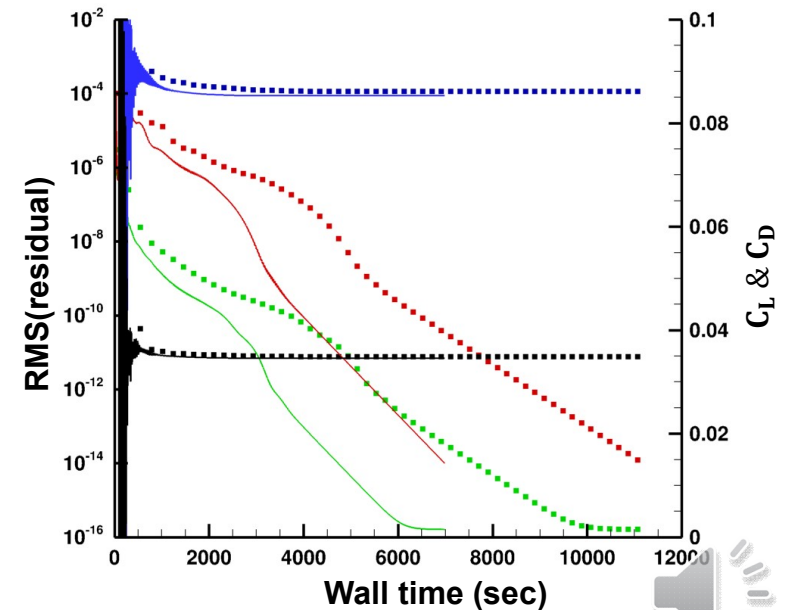
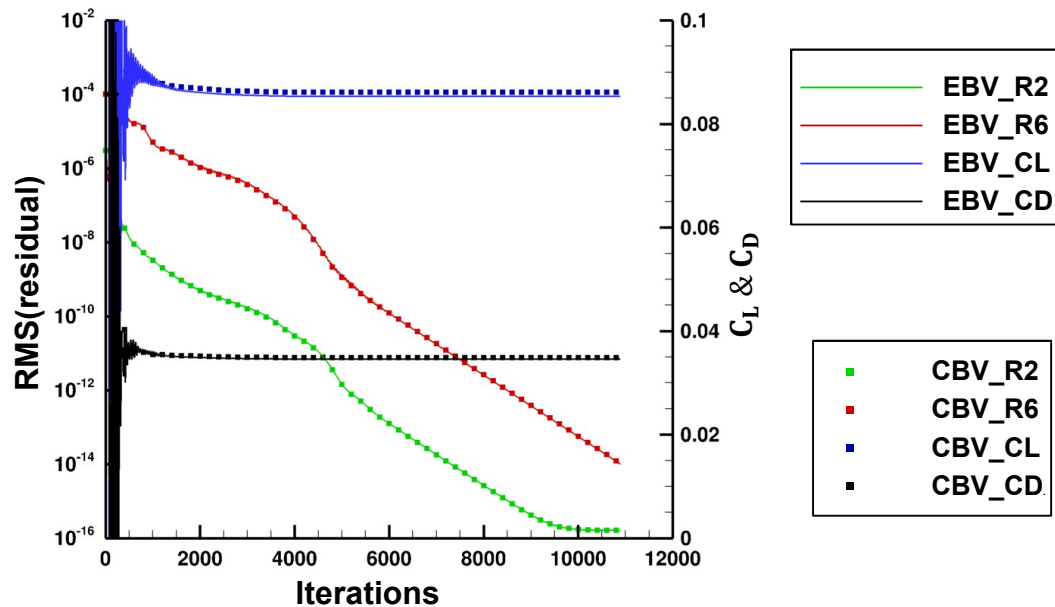
## Wall time and speedup

Computations	CBV time (sec.)	EBV time (sec.)	EBV speedup
Meanflow viscous fluxes	0.050	0.011	78%
Meanflow viscous-flux Jacobian	0.236	0.070	71%
SA-neg diffusion	0.046	0.004	91%
SA-neg diffusion Jacobian	0.124	0.007	95%
Linear solver	0.699	0.424	39%
Iteration with Jacobian update	1.414	0.794	44%
Iteration without Jacobian update	0.944	0.609	36%
Converged solution	11,146	6,959	38%

## Memory usage

CBV (MB)	EBV (MB)	EBV saving
160,048	153,041	4.4%

- EBV memory saving on PH0 grid (71M grid points) is 9%
- Similar EBV speedup is observed for HANIM iterations



# High-Enthalpy, Chemically Reacting, Hypersonic Flow around Blunt Body

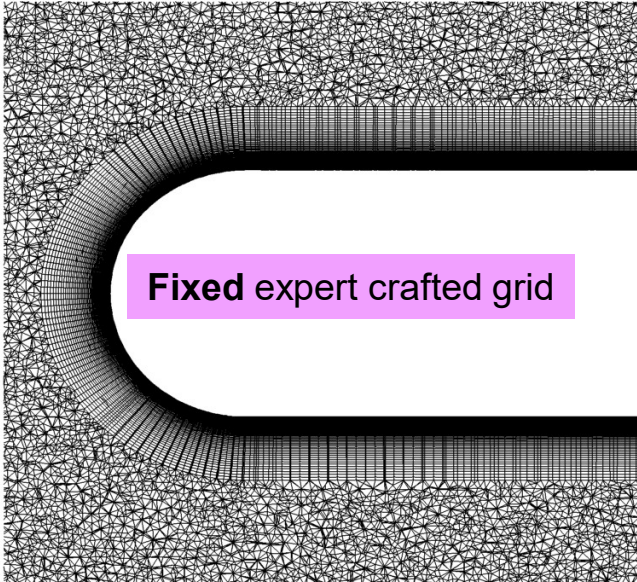
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- laminar flow  $T_\infty = 450$  K,  $p_\infty = 230$  Pa,  $M_\infty = 9.8$ ,  $\alpha = 0^\circ$ ,  $Re_D \approx 14,000$
- noncatalytic wall with constant temperature  $T = 555.5$  K
- standard-air mass fractions at freestream
- five species ( $N_2, O_2, NO, N, O$ ) and five reactions
- two-temperature model

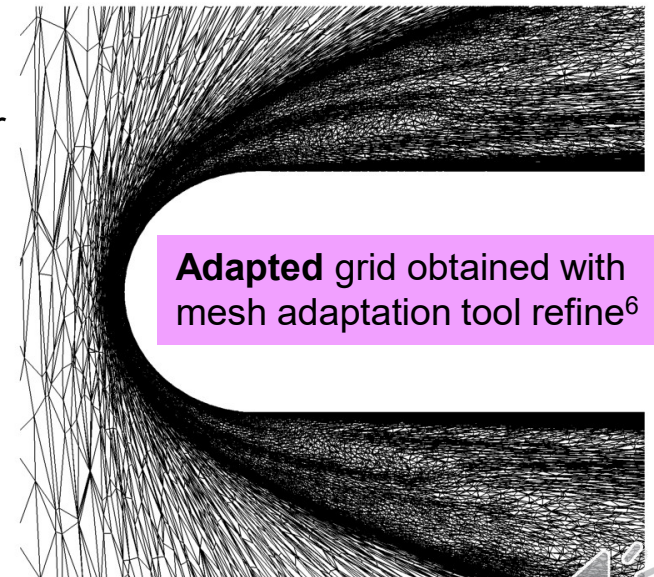
Grids for high-enthalpy flow

Grid	Points	Tetrahedra	Prisms	CPU cores
Fixed	3,459,137	6,639,496	4,627,800	120
Adapted	4,567,239	23,562,449	1,028,320	160



**Fixed** expert crafted grid

- HLLE++ Riemann solver
- van Albada flux limiter
- baseline implicit solver
- coupled Jacobian
- maximum CFL=5



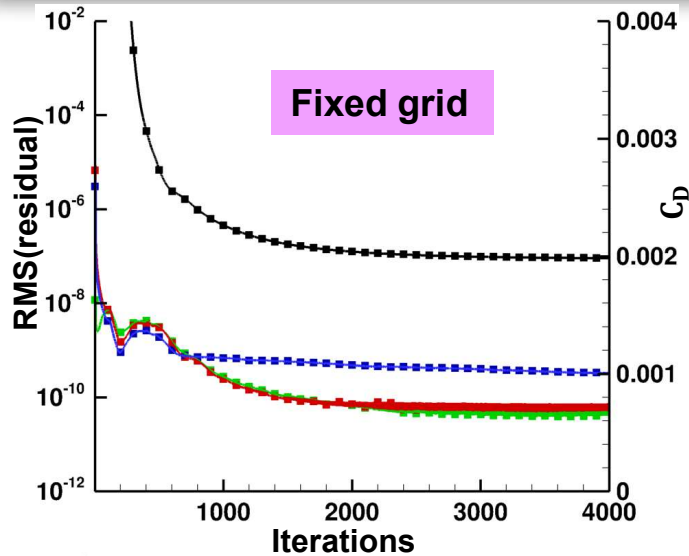
**Adapted** grid obtained with mesh adaptation tool refine<sup>6</sup>

<sup>6</sup>Park M. A. and Carlson J.-R. "Turbulent Output-Based Anisotropic Adaptation," AIAA 2010-0168

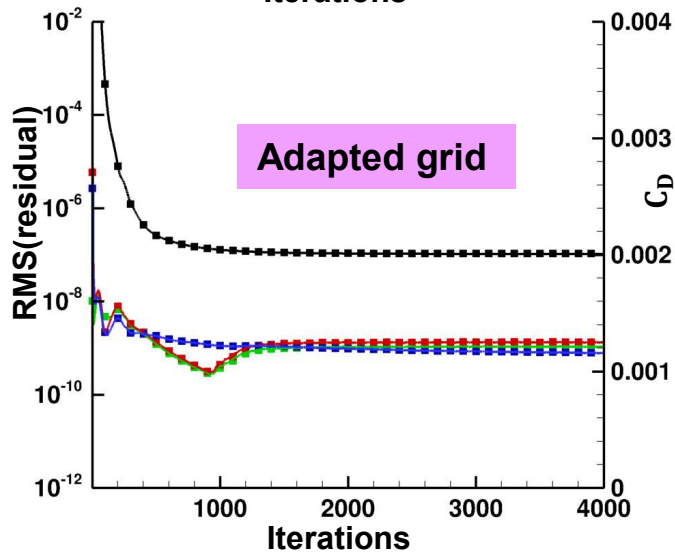
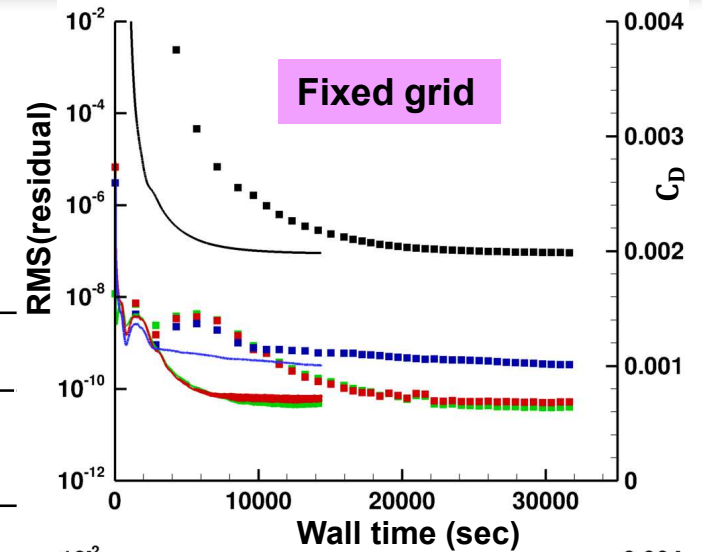




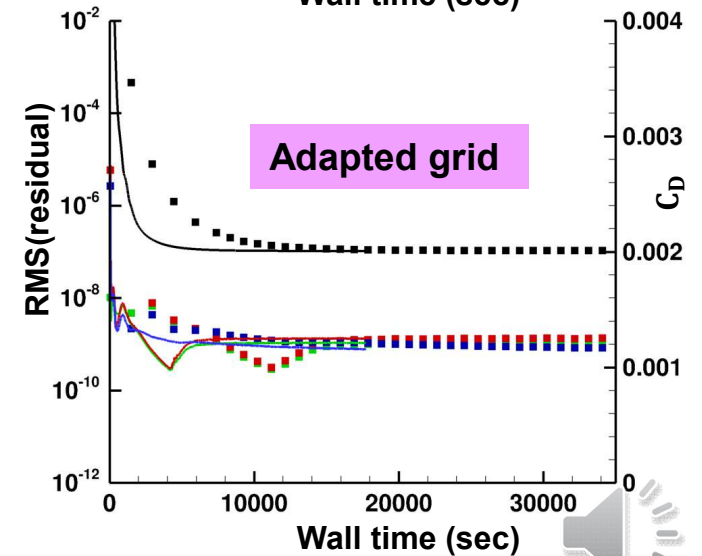
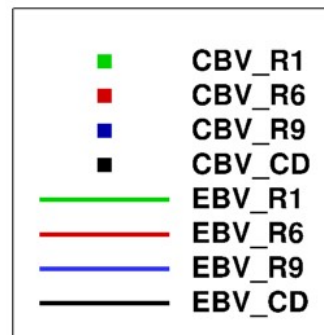
# Iterative Convergence for High-Enthalpy, Chemically Reacting, Hypersonic Flow



	Drag coefficient	
	Fixed	Adapted
CBV	0.0019845	0.0020103
EBV	0.0019834	0.0020087



Significant EBV speedup  
on fixed and adapted grids



# EBV Performance High-Enthalpy, Chemically Reacting, Hypersonic Flow



## Fractions of viscous-kernel and linear-solver computations for high-enthalpy flow

Grids	Viscous method	Viscous flux fraction	Jacobian		Linear solver fraction
			Updates	Fraction	
<b>Fixed</b>	CBV	2.2%	1,668	54.0%	14.7%
	EBV	1.2%	1,671	5.7%	26.0%
<b>Adapted</b>	CBV	1.5%	2,246	58.6%	9.4%
	EBV	1.0%	2,238	6.0%	20.3%

- Viscous Jacobian dominates CBV solver – fraction 54%-59%
- EBV reduces fraction of viscous Jacobian to less than 6% (speedup factor of 20)
- Linear solver speedup is 21% on fixed grid and 6.4% on adapted grid

## Time and speedup

Grid	CBV (sec)	EBV (sec)	EBV speedup
<b>Fixed</b>	32,269	14,350	55.5%
<b>Adapted</b>	40,721	17,686	56.6%

## Memory usage

Grid	CBV (Mb)	EBV (Mb)	EBV saving
<b>Fixed</b>	78,217	74,578	4.7%
<b>Adapted</b>	96,411	101,340	-5.1%

- EBV speedup is above 55.5% (factor of 2.25) on both grids
- EBV saves memory on fixed grid, uses more memory on adapted grid dominated by tetrahedra



# Summary



- **EBV method derived for general mixed-element grids and implemented in practical unstructured-grid, node-centered, finite-volume flow solver**
- **EBV method verified and profiled for a benchmark flow using RANS formulation on family of prismatic-hexahedral grids**
- **EBV method demonstrated for**
  - RANS solutions for benchmark subsonic separated flow
    - ✓ EBV speedup is **37.6% (factor of 1.6) for baseline solver** and **36% (factor of 1.6) for HANIM** (shown in the paper for the baseline iterations on three grids and HANIM iterations on PH2)
    - ✓ EBV **memory saving** is up to **9%**
  - NASA juncture flow model (shown in the paper)
    - ✓ EBV speedup is **55% (factor of 2.3) for baseline solver** and **60% (factor of 2.5) for HANIM**
  - high-enthalpy, chemically reacting, hypersonic flow
    - ✓ EBV speedup is **55.7% (factor of 2.26) on fixed grid** and **57.5% (factor of 2.35) on adapted grid**
    - ✓ EBV memory use is 5% less on fixed grid and 5% more on adapted grid dominated by tetrahedra



# Acknowledgements



Following NASA projects supported this work:

- Transformative Tools and Technologies (TTT) project of Transformative Aeronautics Concepts Program (TACP)
- Revolutionary Vertical Lift Technology (RVLT) project of Advanced Air Vehicles Program (AAVP)
- The first three authors are supported by the NASA Langley Research Center under the cooperative agreement 80LARC17C0004 with the National Institute of Aerospace
- Dr. H. Nishikawa gratefully acknowledges support from the U.S. Army Research Office under the contract/grant number W911NF-19-1-0429.

Computations were performed on NASA LaRC K-Cluster

Authors would like to thank Dr. Balaji Venkatachari and Dr. Mike Park for reviewing the paper and providing helpful comments





# Thank You!





# Back Up

# CBV Formulation on Tetrahedron



Green-Gauss gradient evaluation on a tetrahedron:

$$\nabla^h \varphi = \frac{(\varphi_2 + \varphi_3 + \varphi_4)\mathbf{n}_1 + (\varphi_1 + \varphi_3 + \varphi_4)\mathbf{n}_2 + (\varphi_1 + \varphi_2 + \varphi_4)\mathbf{n}_3 + (\varphi_1 + \varphi_2 + \varphi_3)\mathbf{n}_4}{3 Vol}$$

$\mathbf{n}_i$  is outward directed area vector of face opposite to  $\mathbf{p}_i$

$$\mathbf{n}_1 = -(\mathbf{n}_2 + \mathbf{n}_3 + \mathbf{n}_4)$$

Edge-based difference operator:  $\Delta_{ij}\varphi \equiv \varphi_i - \varphi_j$

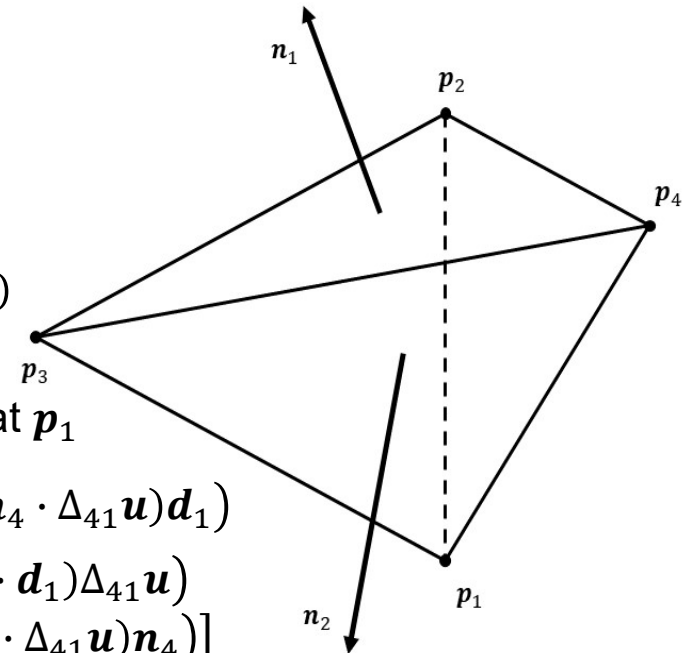
Edge-based gradient:  $\nabla^h \varphi = \frac{-1}{3 Vol} (\Delta_{21}\varphi \mathbf{n}_2 + \Delta_{31}\varphi \mathbf{n}_3 + \Delta_{41}\varphi \mathbf{n}_4)$

Viscous-flux edge-based contributions to momentum residual at  $\mathbf{p}_1$

$$\begin{aligned} \mathbf{R}_{m1} = \mathbf{R}_{m1} + \frac{\mu + \mu_t}{3 Vol} \left[ -\frac{2}{3} \left( (\mathbf{n}_2 \cdot \Delta_{21}\mathbf{u})\mathbf{d}_1 + (\mathbf{n}_3 \cdot \Delta_{31}\mathbf{u})\mathbf{d}_1 + (\mathbf{n}_4 \cdot \Delta_{41}\mathbf{u})\mathbf{d}_1 \right) \right. \\ \left. + ((\mathbf{n}_2 \cdot \mathbf{d}_1)\Delta_{21}\mathbf{u} + (\mathbf{n}_3 \cdot \mathbf{d}_1)\Delta_{31}\mathbf{u} + (\mathbf{n}_4 \cdot \mathbf{d}_1)\Delta_{41}\mathbf{u}) \right. \\ \left. + ((\mathbf{d}_1 \cdot \Delta_{21}\mathbf{u})\mathbf{n}_2 + (\mathbf{d}_1 \cdot \Delta_{31}\mathbf{u})\mathbf{n}_3 + (\mathbf{d}_1 \cdot \Delta_{41}\mathbf{u})\mathbf{n}_4) \right] \end{aligned}$$

$\mathbf{u}$  is velocity vector

$\mathbf{d}_i = \frac{1}{3} \mathbf{n}_i$  is directed area vector of dual-volume surface within tetrahedron associated with  $\mathbf{p}_i$



CBV formulation is not edge-based because viscosity is evaluated at cell centroid

# EBV Formulation on Tetrahedron



Viscosity evaluated at edge

$$(\mu + \mu_t)_j = \frac{(\mu + \mu_t)_j + (\mu + \mu_t)_i}{2}$$

Viscous-flux contributions to momentum residual at  $\mathbf{p}_1$  from edge  $[\mathbf{p}_1, \mathbf{p}_2]$

$$\mathbf{R}_{m1} = \mathbf{R}_{m1} + \frac{(\mu + \mu_t)_2}{9 Vol} \left[ (\mathbf{n}_2 \cdot \mathbf{n}_1) \mathbf{I} - \frac{2}{3} \mathbf{n}_1 \mathbf{n}_2^T + \mathbf{n}_2 \mathbf{n}_1^T \right] \Delta_{21} \mathbf{u}$$

$3 \times 3$  matrix represents nine EBV coefficients collected over all tetrahedra ( $i = 1, 2, 3, \dots$ ) that share edge  $[\mathbf{p}_1, \mathbf{p}_2]$

$$\mathbf{C}_{2,1} = \sum_i \frac{1}{9 Vol_i} \left[ (\mathbf{n}_{i2} \cdot \mathbf{n}_{i1}) \mathbf{I} - \frac{2}{3} \mathbf{n}_{i1} \mathbf{n}_{i2}^T + \mathbf{n}_{i2} \mathbf{n}_{i1}^T \right]$$

- EBV matrix is symmetric for interior edges
- six EBV coefficients for each interior edge
- nine EBV coefficients for each boundary edge

# EBV Source Terms

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$x$ -momentum:  $\frac{5}{6} [(\partial_y v)(\partial_x \mu) - (\partial_y \mu)(\partial_x v) + (\partial_z w)(\partial_x \mu) - (\partial_z \mu)(\partial_x w)]$

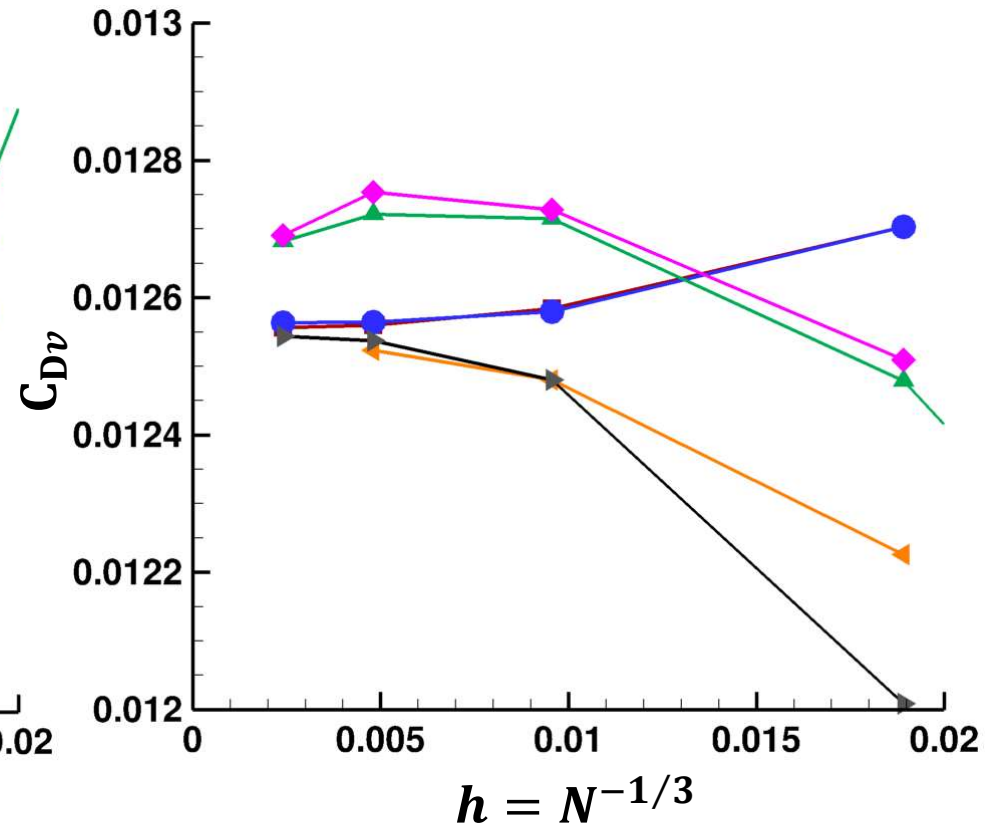
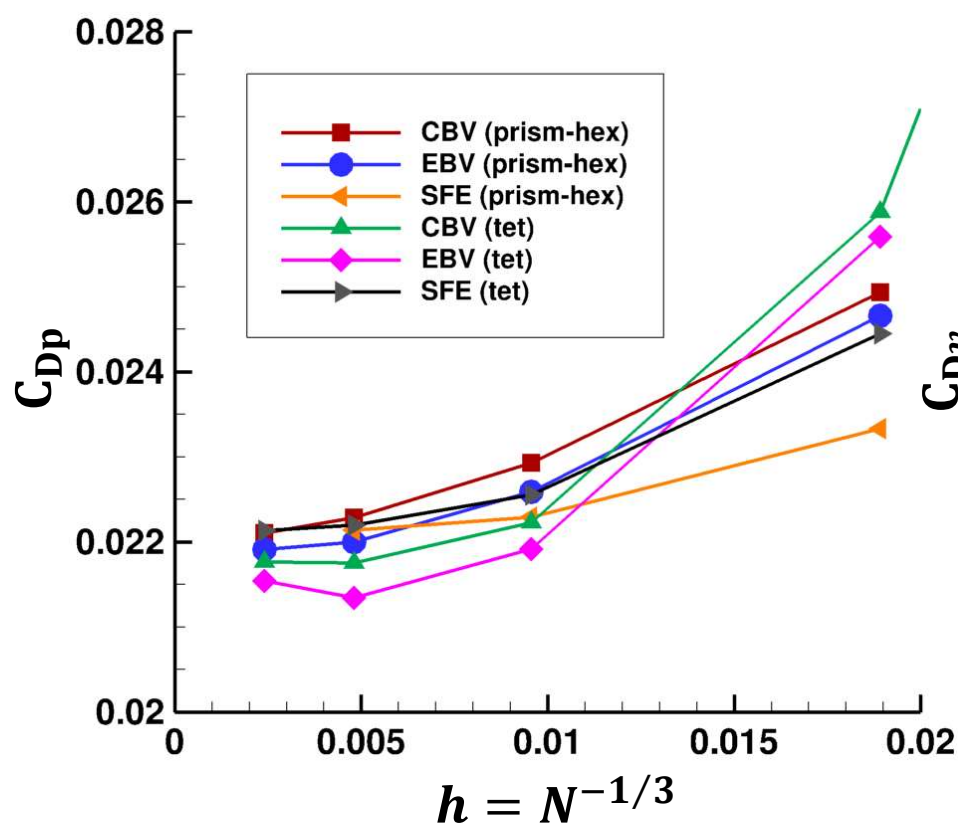
$y$ -momentum:  $\frac{5}{6} [(\partial_x u)(\partial_y \mu) - (\partial_x \mu)(\partial_y u) + (\partial_z w)(\partial_y \mu) - (\partial_z \mu)(\partial_y w)]$

$z$ -momentum:  $\frac{5}{6} [(\partial_x u)(\partial_z \mu) - (\partial_x \mu)(\partial_z u) + (\partial_y v)(\partial_z \mu) - (\partial_y \mu)(\partial_z v)]$

Energy:  $\frac{5}{6} \left\{ 2\mu \left[ (\partial_x u)(\partial_y v) + (\partial_x u)(\partial_z w) + (\partial_y v)(\partial_z w) \right] \right.$   
 $\left. - \left( (\partial_y u)(\partial_x v) + (\partial_z u)(\partial_x w) + (\partial_z v)(\partial_y w) \right) \right]$   
 $+ (\partial_x \mu) [u(\partial_y v + \partial_z w) - v(\partial_y u) - w(\partial_z u)]$   
 $+ (\partial_y \mu) [v(\partial_x u + \partial_z w) - u(\partial_x v) - w(\partial_z v)]$   
 $+ (\partial_z \mu) [w(\partial_x u + \partial_y v) - u(\partial_x w) - v(\partial_y w)] \}$



# Drag Coefficients with EBV on Tetrahedral Grids





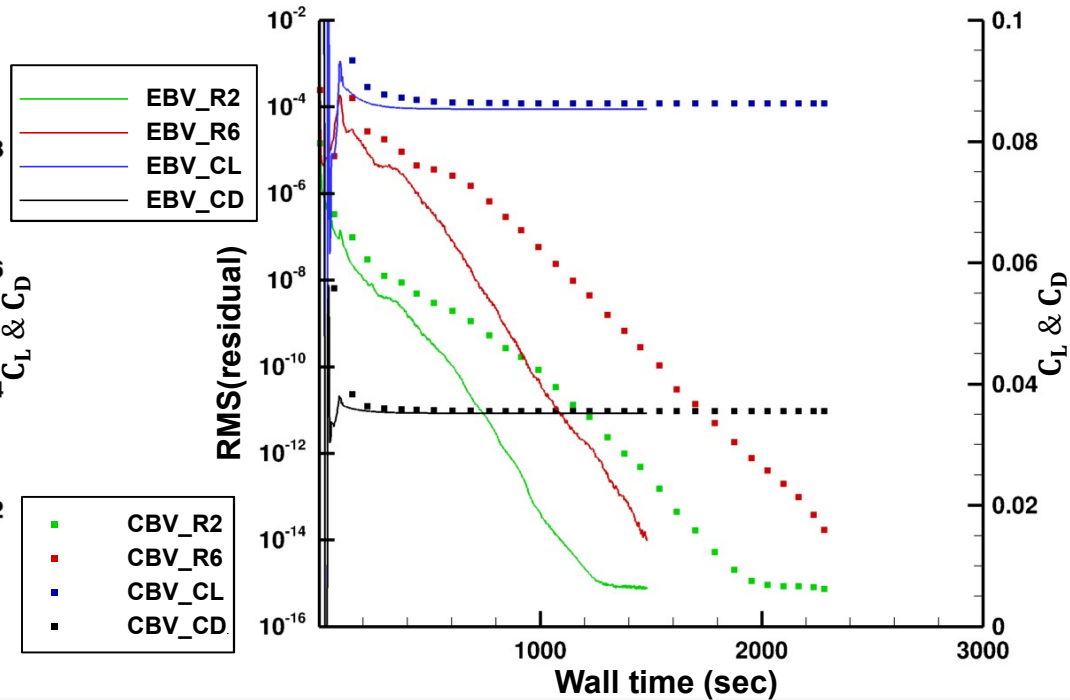
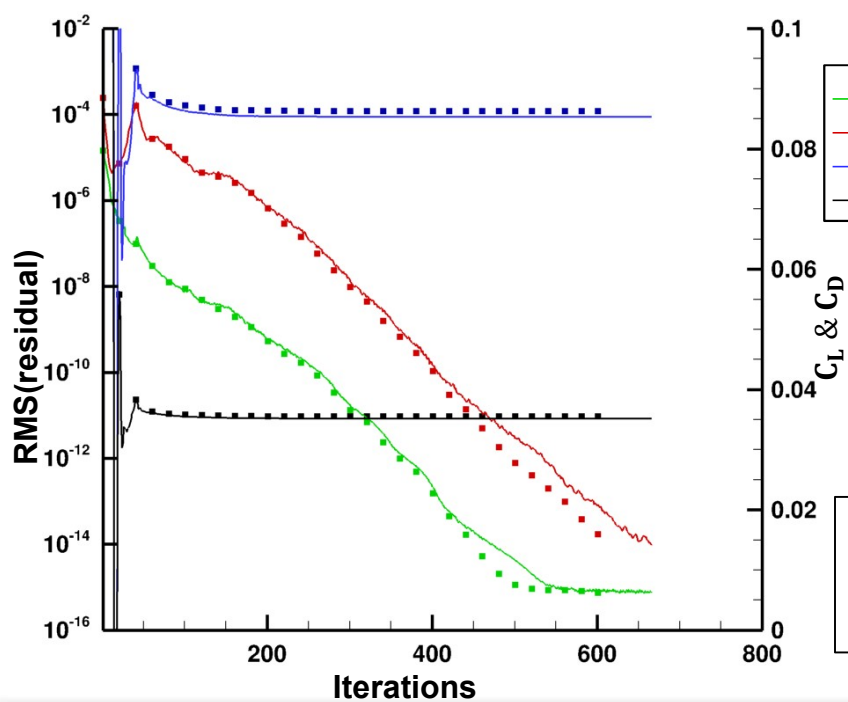
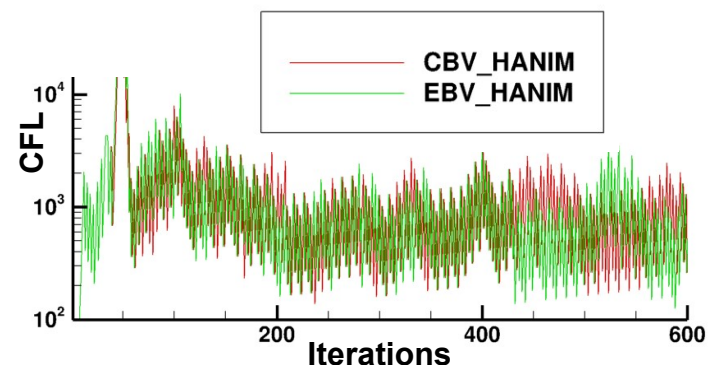
# HANIM Solutions on PH2 Grid (1.1M Grid Points)

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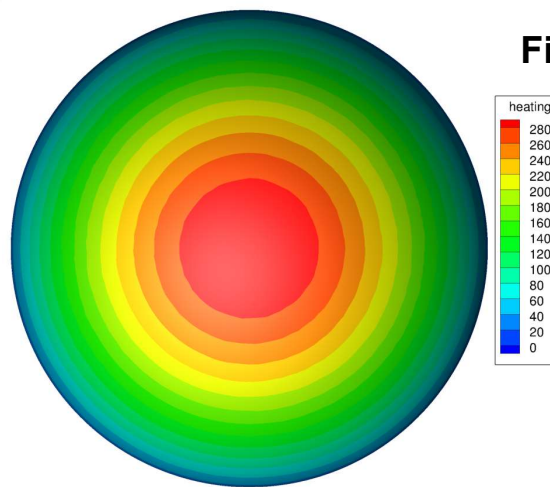


## Nonlinear iterations

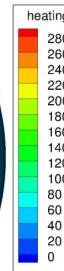
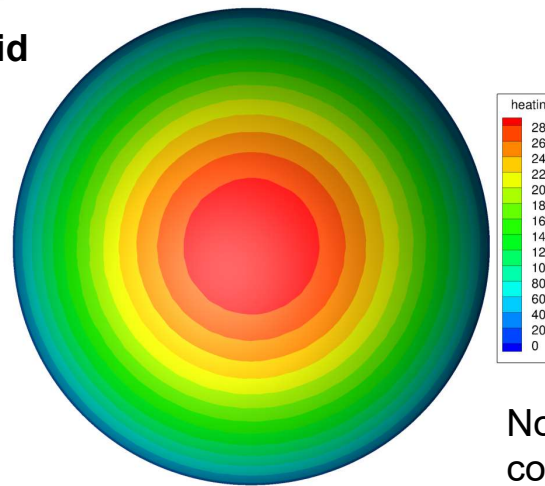
	Baseline iterations		HANIM iterations		HANIM speedup
	Iterations	Wall time (sec)	Iterations	Wall time (sec.)	
CBV	4,005	3,782	618	2,311	39%
EBV	4,026	2,372	666	1,478	38%
EBV speedup		37.3%		36.1%	



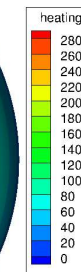
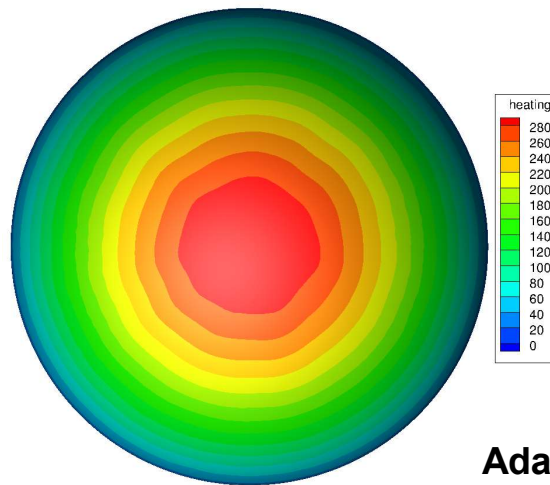
# Surface Heating for High-Enthalpy, Chemically Reacting, Hypersonic Flow



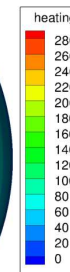
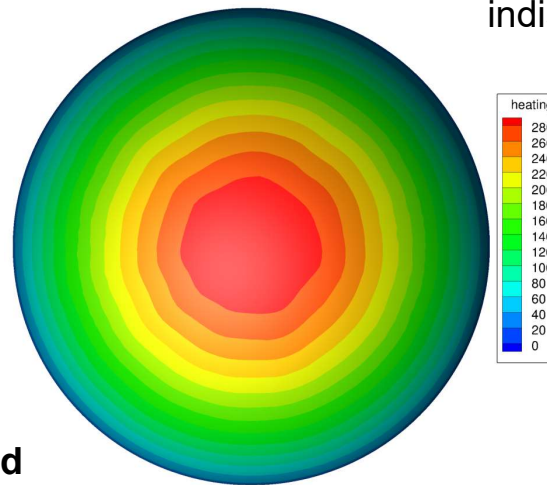
**Fixed grid**



**CBV solutions**



**EBV solutions**



Noticeable differences between contours computed on fixed and adapted grids

EBV and CBV contours are indistinguishable on the same grids

**Adapted grid**